# Determination of Orbital Parameters by Use of an Astrogyrodynamics Concept

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**An astrogyrodynamic concept is derived which couples the dynamics of a gyroscope with orbit equations. It is shown that, by varying the gyro spin rate and precession torque to satisfy the coupled equations, the resulting spin rate will also be a cosine function about a mean value. It is further shown that the amplitude of the gyro spin rate variation is proportional to the orbit eccentricity, the period of the spin rate variation being proportional to the orbital period. A block diagram is presented which demonstrates that an error signal may be formulated and that the astrogyro spin rate and precession torque may be incorporated as a servoloop. Error derivatives and curves for a given orbit are presented. To minimize random and systematic errors, a least-squares technique is developed, and it is shown that, by making observations at predetermined intervals, the best estimate of the parameters in a least-squares sense is always obtained. An orbit is postulated and system performance, when certain errors are present, is evaluated.**

## Nomenclature

$a$ = semimajor axis length, miles
$c_1$ = areal velocity constant, miles²/sec
$e$ = eccentricity
$H$ = gyro angular momentum, ft-lb-sec
$H_M$ = gyro mean angular momentum, ft-lb-sec
$I$ = moment of inertia, ft-lb-sec²
$K_\theta$ = $\ddot{\theta}$ error coefficient
$N$ = number of observations
$P$ = orbital period, sec
$T$ = torque, ft-lb
$t$ = time, sec
$V$ = speed, miles/sec
$\alpha$ = $a(1 - e^2)$, semilatus rectum, miles
$\beta$ = amplitude coefficient of rotor speed variation
$\gamma$ = gyro speed variation angle, deg
$\Delta\theta$ = observation interval, deg
$\epsilon_\omega$ = speed control error signal, rad/sec
$\epsilon_T$ = line of sight error, deg
$\tau_\omega$ = spin rate time lag, sec
$\tau$ = equivalent rate system time constant, sec
$\dot{\phi}$ = gyro precession rate, rad/sec
$\phi$ = gyro precession angle measured from $t = 0$, deg
$\Phi$ = gyro speed phase lag, deg
$\omega$ = gyro speed, rad/sec
$\Omega$ = gyro speed difference, rad/sec
$\omega_M$ = mean gyro speed
$\omega_0$ = initial

***Subscripts***

0 = initial value
$c$ = computed value

## Introduction

THE geometry of the orbit of a body in a central force field has been given extensive attention in theoretical and celestial mechanics. However, when one attempts to apply the theory to actual conditions, one confronts the problem of data acquisition and synthesis and finds that many of the instruments with desirable physical properties are not available,[1-3] because development of instrumentation for space navigation has not kept pace with theoretical work. Indeed, space systems would be simplified if instrumentation were available whose natural characteristics were such that the many constants appearing in theoretical work could be "measured" directly. However, natural laws as well as undiscovered simpler methods preclude the use of simple, accurate, and reliable instruments for space navigation aids.

The motion of an orbiting body is characterized by an ellipse whose properties are determined by its eccentricity, semimajor axis length, and the position and length of the radius vector from the focus (center of planet). Under certain conditions it is desirable to obtain these parameters by use of an onboard, self-contained, nonradiating, low-power system, i.e., one in which all sensing elements are integral to the system and none employs an active electromagnetic energy radiating device. In the future, systems of this type will play the same role in space travel that inertial systems do in the guidance of long-range ballistic missiles and aircraft. A spacecraft will be able to determine its orbit without recourse to earth or planet-bound observation and data acquisition sensor. A projection of the probable trends in space travel leads one to the conclusion that studies on this problem should begin now.

At present, orbits are primarily being computed by use of range and range-rate data, because radar was at an advanced stage of development at the dawn of the space age. Extension of range capability was readily accomplished because, with all instrumentation, earthbound, size, weight, and power requirements were of secondary importance. However, when efforts are made to put these sensors into the spacecraft, the importance of these factors is drastically increased. Thus one is tempted to seek other types of sensors and computations for orbit determination.

In the astrogyrodynamics system presented herein, radar (or other ranging system) is completely eliminated and all data sensors are centralized in the spacecraft. Relative to an observer on the surface of the earth, six parameters are required to establish the position of a spacecraft. Three of these may be classed as inertial elements in that they are independent of the position of the observer. It is the task of the astrogyrodynamic system to obtain these three inertial elements without relying upon an earth- or planet-bound observer. The purpose of this paper is to introduce the use of an astrogyrodynamic concept and to present a method that could be used to implement or aid a space guidance system while performing a specific task. The following assumptions are made: 1) the earth is spherical, 2) the vehicle flies through

Presented as Preprint 64-30 at the AIAA Aerospace Sciences Meeting, New York, January 20–22, 1964; revision received August 31, 1964. Computations were made at the Bell Aerosystems Computer Center. The author would like to express his appreciation to P. Ott for preparation of the digital program and F. Grochowiak for aid in preparation and proofing of transcript.

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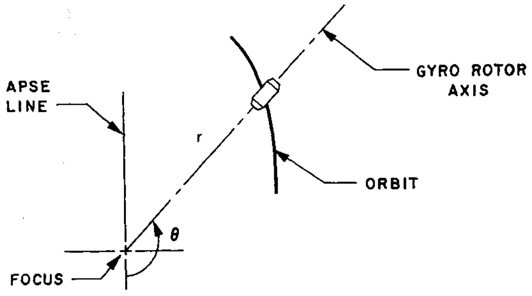


**Fig. 1 Gyro and orbit configuration.**

a perfect vacuum, and 3) the central force field is due to two bodies alone.

## Similarity Concept

The idealized relationship between torque and precession rate of a single-degree-of-freedom gyroscope is given by

$$T = H\dot{\phi} \tag{1}$$

In many applications to present day problems, the gyroscope angular momentum $H$ is necessarily constant. If, however, physical conditions are such that

$$H = H_M[1 + \beta\cos(\gamma + \Phi)] \tag{2}$$

then Eq. (1) may be written in nondimensional form

$$H_M\dot{\phi}/T = 1/[1 + \beta\cos(\gamma + \Phi)] \tag{3}$$

It has been shown that a body in orbit about a planet will transverse a path whose conic equation is given by

$$r = \alpha/(1 + e\cos\theta) \tag{4}$$

subject to the restraint

$$r^2\dot{\theta} - C_1 \tag{5}$$

Equation (4) in nondimensional form becomes

$$r/a(1 - e^2) = 1/(1 + e\cos\theta) \tag{6}$$

Since Eqs. (3) and (6) are nondimensional and of identical character, the following equalities can be made to hold:

$$H_M\frac{\dot{\phi}}{T} = \frac{1}{1 + \beta\cos(\gamma + \Phi)} = \frac{r}{\alpha} = \frac{1}{1 + e\cos\theta} \tag{7}$$

subject to the conditions

$$\theta = \gamma + \Phi \quad \text{and} \quad \beta = e \tag{8}$$

But from Eq. (3) we note that $\beta$ is the amplitude of the cosine variation of the gyroscope rotor angular momentum. This condition implies that the eccentricity of an elliptic, or general conic section, orbit may be determined by properly torquing a gyroscope and at the same time varying its angular momentum according to Eq. (2).

Assume that the gyrorotor moment of inertia is constant and the angular velocity is allowed to vary. Then since

$$H = I\omega \qquad H_M = I\omega_M \tag{9}$$

Eq. (3) may be written as

$$\omega = \omega_M[1 + \beta\cos(\gamma + \Phi)] \tag{10}$$

If Eqs. (8) are satisfied, and physical conditions are such that $\dot{\phi} = \dot{\theta}$ (which will be discussed in detail later), squaring Eq. (7), using Eq. (5), and eliminating $\dot{\phi}$ yields

$$\alpha^2T/H_M[1 + \beta\cos(\gamma + \Phi)]^3 = C_1 \tag{11}$$

Since this equation is constant for all $T$ and corresponding $\gamma$ we have

$$\frac{T}{T_1} = \frac{[1 + \beta\cos(\gamma + \Phi)]^3}{[1 + \beta\cos(\gamma_1 + \Phi)]^3} \tag{12}$$

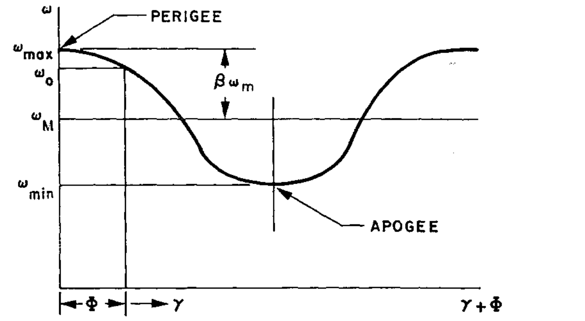


**Fig. 2 Gyro spin rate vs phase angle.**

Using Eq. (10) in Eq. (12) yields

$$T/T_1 = (\omega/\omega_1)^3 \tag{13}$$

which may also be written as

$$\dot{\phi}/\dot{\phi}_1 = (\omega/\omega_1)^2 \tag{14}$$

If Eqs. (13) and (14) are satisfied, then Eq. (7) is satisfied. To impose the restraint $\dot{\phi} - \dot{\theta}$, assume that the body in orbit around the planet contains the gyro (astrogyro) whose spin axis is constrained to remain aligned with (i.e., to track) the planet's center, or local vertical; Fig. 1 illustrates this. Figure 2 shows the variation of $\omega$ during a complete orbit, and describes the function $\omega = f(\gamma + \Phi)$, but since all restraints are satisfied, it may be interpreted as being $\omega = f(\theta)$.

## Physical System

A physical system capable of serving as an effective aid to implement the previously derived equations will now be postulated. The gyrotorque must be applied in accordance with Eq. (13), whereas the rotor angular velocity variation is governed by Eq. (14). If the spin axis is initially aligned with the local vertical, this alignment will be maintained as long as the two previously mentioned conditions are fulfilled.

A command signal for gyro speed regulation is obtained in the following manner. Write Eq. (14) as

$$[(\omega_0 + \Delta\omega)/\omega_0]^2 = \dot{\phi}/\dot{\phi}_0 \tag{15}$$

Expanding and simplifying Eq. (14) to obtain a form for a control signal in the gyro speed control loop gives

$$\Delta\omega(2 + \Delta\omega/\omega_0) - \omega_0(\dot{\phi}/\dot{\phi}_0 - 1) = \epsilon_\omega \tag{16}$$

Figure 3 shows a block diagram of a proposed system. Planetary tracking begins when switch $s_1$ is closed and $s_2$ is opened, causing a change in $\omega$ and $T$, which drives $\epsilon_\omega$ to zero. A tracking procedure may now be stated as follows:

1) With $\omega$ constant, and the gyro precession axis normal to the orbital plane, align the spin axis with the planet's center

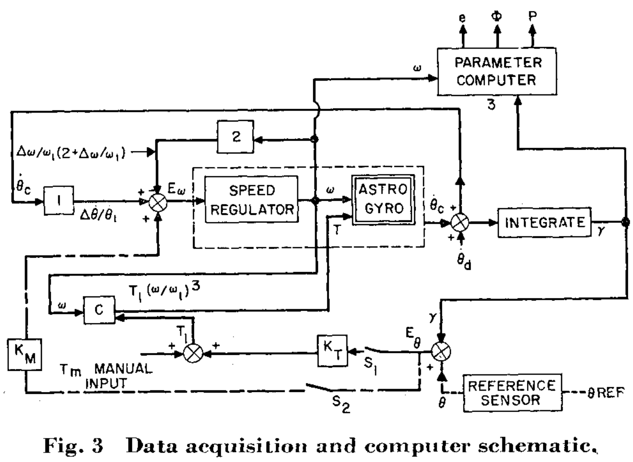


**Fig. 3 Data acquisition and computer schematic.**

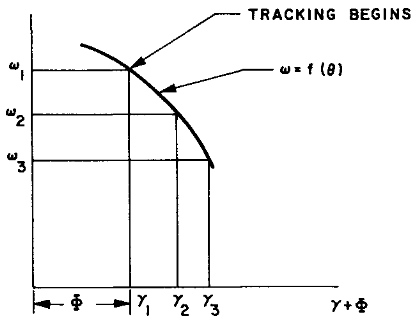


**Fig. 4 Gyro spin rate vs phase angle.**

by torquing the gyro (manually, programed, or remote control).

2) With the torque constrained according to Eq. (13), vary $\omega$ in accordance with Eq. (14).

3) Obtain $\gamma$ by integrating $\dot{\gamma} = \dot{\phi}$ (or equivalent).

4) The corresponding $\omega$ vs $\gamma$ curve is shown in Fig. 2. Step 2 began at an arbitrary position designated by the phase angle $\Phi$.

5) Continue to track until sufficient data recorded for memory are acquired to evaluate Eq. (2) (or for a complete orbit if possible).

6) The maximum amplitude of the resulting function (Fig. 2) is proportional to $e$ and the instantaneous value of $(\gamma + \Phi)$ equals $\theta$, the geocentric angle measured from perigee.

Computer 1 is to acquire the value of $\omega = \omega_0$ at instant switch is closed ($t = 0$); compute the ratio $\dot{\phi}/\dot{\phi}_0$, where $\dot{\phi}_0$ is the value of $\dot{\phi}$ at $t = 0$, and compute the value of the function shown at the exit of computer 1. Computer 2 is to acquire the initial value $\omega_0$, the subsequent variation from the value $\Delta\omega$, and formulate the function shown. Computer 3 of Fig. 3 receives $\omega(t)$ and $\gamma(t)$; since these outputs satisfy Eq. (7), Eq. (10) gives the required variation of the function $\omega = f(\gamma)$.

A portion of Eq. (10), obtained by tracking the planet or local vertical for an extended interval of time, is shown graphically in Fig. 4. Since there are three unknowns, $\omega_M$, $\beta$, $\Phi$, three independent measurements yielding three independent equations will be required to obtain a solution. These equations may be written as

$$\omega_n = \omega_M[1 + \beta \cos(\gamma_n + \Phi)] \tag{17}$$

where $n = 1, 2, 3$ (see Fig. 4) and $\gamma_1 = 0$. Solving Eq. (18) yields

$$\tan\Phi = q_1 \frac{(\Omega_1/\Omega_2) - q_2}{(\Omega_1/\Omega_2) + q_3} \tag{18}$$

$$f(\Phi) = p(\omega_1 - \omega_2)/(K_1 + \tan\Phi) \tag{19}$$

$$\omega_m = \omega_1 - f(\Phi) \tag{20}$$

$$\beta = e = \frac{f(\Phi)}{\omega_1 - f(\Phi)} (1 + \tan^2\Phi)^{1/2} = \frac{f(\Phi)}{\omega_M \cos\Phi} \tag{21}$$

where

$q_1 = (\cos\gamma_2 - \cos\gamma_3)/(\sin\gamma_2 - \sin\gamma_3)$
$q_2 = (1 - \cos\gamma_2)/(\cos\gamma_2 - \cos\gamma_3)$
$q_3 = \sin\gamma_2/(\sin\gamma_2 - \sin\gamma_3)$
$K_1 = (1 - \cos\gamma_2)/\sin\gamma_2$
$p = 1/\sin\gamma_2$
$\Omega_1 = \omega_1 - \omega_2 \qquad \Omega_2 = \omega_2 - \omega_3$

To simplify computer design, $q_n$, $K_1$, and $p$ may be "built-in" constants. This is accomplished by monitoring the input such that the values of $\omega_n$ are taken at preset intervals of $\gamma$. For example, $\gamma_3 = \lambda\gamma_2$, where $\lambda$ may be a suitable preassigned number. Preliminary computations indicate an improvement in accuracy for values of $\gamma_n > 1°$.

By measuring the gyrorotor frequency, and for the special case presented in Table 1, Eq. (18) may be used to compute $\Phi$, which now represents the magnitude of the geocentric angle measured from perigee at initiation of tracking. Equation (27) determines the eccentricity of the ellipse. Subsequently, $r/a$ may be obtained from Eq. (6).

The orbital elements $e$, $\theta r/a$ are now determined. The ellipse (or trajectory) relative to inertial space will be explicitly known if at least one range measurement is made. The orbit period may be computed from the formula

$$P = \frac{2\pi}{(\omega_m/\omega_0)^2\dot{\theta}_0(1 - e^2)^{3/2}} \tag{22}$$

The semimajor axis length $a$ is obtained from

$$P = 2\pi(a^3/\mu)^{1/2} \tag{23}$$

The geocentric range $r = f(\theta)$ may subsequently be obtained by using $\omega = f(\theta)$ to form $r = f(\omega)$. In any subsequent orbit position, geocentric range as a function of geocentric angle will be available.

## Data Smoothing and Error Analysis

Since all measured data is contaminated with noise, a smoothed value of $e$ may be obtained by subdividing each $\gamma_n$ interval into $k$ parts and computing $e$ for intervals

$$\gamma_{3+nk} = \lambda\gamma_{2+nk} \qquad n = 1, 2, 3 \ldots \tag{24}$$

where $K$ is a convenient subinterval. The computer will accept and store $\gamma$ for each subinterval $k$. The subinterval duration may be made constant and "built-in" the computer and consequently is available for computing the smoothed value of $e$. The astrogyro error derivatives were obtained and are given as follows:

$$\partial e/\partial\omega_m = -e/\omega_m$$

$$\partial e/\partial\Omega_1 = e/\Omega_1$$

$$\partial e/\partial\Phi = (1 - K_1 \tan\Phi)e/(K_1 + \tan\Phi)$$

$$\frac{\partial\Phi}{\partial\Omega_1} = \frac{1}{\gamma_1\Omega_2} \frac{\gamma_3 + \gamma_2}{[(\Omega_1/\Omega_2) - \gamma_2]^2} \sin^2\Phi$$

$$\partial\Phi/\partial\Omega_2 = -(\Omega_1/\Omega_2)\partial\Phi/\partial\Omega_1$$

$$\partial\Omega_1/\partial\dot{\phi}_2 = -\omega_1/2(\dot{\phi}_1\dot{\phi}_2)^{1/2}$$

$$\partial\Omega_1/\partial\dot{\phi} = -(\dot{\phi}_2/\dot{\phi}_1)\partial\Omega_1/\partial\dot{\phi}_2$$

$$\partial\omega_m/\partial\omega_1 = \omega_m/\omega_1$$

$$\partial\omega_m/\partial_e = -\cos\Phi\omega_1/(1 + e\cos\Phi)^2$$

$$\partial\omega_m/\partial\Phi = e\sin\Phi\omega_1/(1 + e\cos\Phi)$$

$$\partial\Omega_1/\partial\omega_1 = 1 - (\dot{\phi}_2/\dot{\phi}_1)^{1/2}$$

$$\partial\Omega_2/\partial\omega_2 = 1 - (\dot{\phi}_3/\dot{\phi}_2)^{1/2}$$

Obtaining total differentials (only $\Delta e$ is given below), the error in $e$ may be written as

$$\frac{\Delta e}{e} = -\frac{\Delta\omega_m}{\omega_m} + \frac{\Delta\Omega_1}{\Omega_1} - \frac{1 - K_1\tan\Phi}{K_1 + \tan\Phi}\Delta\Phi \tag{25}$$

**Table 1 Orbit computer constants**

| | $\lambda = 2$, $\gamma_2 = 1°$ | $\lambda = 2$, $\gamma_2 = 15°$ |
|---|---|---|
| $q_1$ | −0.02638 | −0.41421 |
| $q_2$ | +0.3263 | 0.34104 |
| $q_3$ | −1.0000 | −1.07314 |
| $K_1$ | 0.00859 | 0.13164 |
| $p$ | 57.2958 | 3.86369 |

Figures 5–7 are plots of error components given by Eq. (25) and show the contribution of each component to the error in eccentricity as a function of the angular observation interval using the following constants:

$$\Omega_1 = 82.03 \qquad \Omega_2 = 82.83$$

$$\Phi = 60° \quad \gamma_1 = 1° \qquad \lambda = 2$$

$$q_1 = -0.0264 \qquad q_2 = 0.3263 \qquad q_3 = -1.0000$$

$$\omega_1 = 20{,}700 \qquad \omega_m = 18{,}000 \qquad \Delta\omega_1 = 0.1$$

## Parameter Determination Employing Least-Squares Technique

Since all observations and data sensing instrumentation are subject to random disturbances, a least-squares theory is now developed. The true trajectory is evaluated by solving the equations of a conic section describing a satellite orbit, i.e., Eqs. (6, 7, 23), and

$$V = \left[\mu\left(\frac{2}{r} \pm \frac{1}{a}\right)\right]^{1/2} \quad \begin{array}{l} + \text{ for } e > 1 \\ - \text{ for } e < 1 \\ a = \infty, e = 1 \end{array} \tag{26}$$

Given $a$, $e$, $C_1$, $\mu$, $\theta_0$, these equations yield, $\theta$, $r$, $V$, $P$, which are actual trajectory functions required to determine present and future satellite position in space. The errors in measurement of geocentric angular rate $\dot{\theta}$ are assumed to be from two sources: 1) random noise and 2) gyro drift (systematic). Thus

$$\dot{\theta} = \dot{\theta}_n + \dot{\theta}_d \tag{27}$$

(The drift component in this case may be assumed to be composed of the gyro drift, platform error, plus satellite motion in the orbital plane.) The noise composition is assumed to be due to operator as well as instrument random errors. The gyro performance is assumed to be characterized by a first-order time lag; consequently, the measured rate $\dot{\theta}_c$ as a function of the actual rate $\dot{\theta}$ is given by

$$\tau_1\ddot{\theta}_c + \dot{\theta}_c = \dot{\theta} \tag{28}$$

where $\tau_1$ equals the equivalent rate system time constant. The corresponding torque becomes

$$T_c = I\omega\dot{\theta}_c \tag{29}$$

The condition for initiation of automatic tracking is determined by the following relations:

$$\dot{\theta}_c = \dot{\theta}_0 \qquad T_c = T_0 \text{ for } t = t_0 \tag{30}$$

Define

$$J = \dot{\theta}_0/\omega_0^2 \qquad L = T_0/\omega_0^3 \tag{31}$$

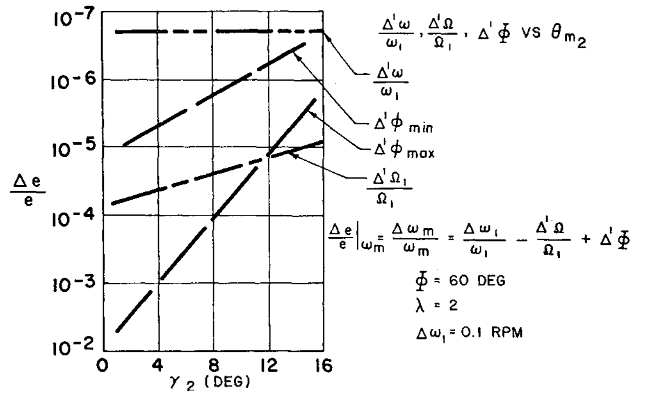


**Fig. 6 Error in eccentricity due to error in gyro speed and speed variation.**

then Eqs. (13) and (14), respectively, become

$$\omega = (\dot{\theta}/J)^{1/2} \tag{32}$$

$$T = L\omega^3 \tag{33}$$

Since $\omega$ in Eq. (32) is obtained by a servocommand loop, a time lag of at least first order will be present. Assuming a first-order lag, the equation becomes

$$\tau_\omega\dot{\omega} + \omega = (\dot{\theta}_c(1 + K_\theta)/J)^{1/2} \tag{34}$$

where $K_\theta$ is introduced as an error parameter. The computed, (or measured), increment of the true anomaly is given by

$$\theta_m = \int_0^\tau \dot{\theta}_c dt \tag{35}$$

being the elapsed time.

Equation (32) is related to the eccentricity and true anomaly by

$$\omega = \omega_m |1 + e_c \cos(\gamma + \Phi)| \tag{36}$$

$\Phi$ is the true anomaly when the tracking sequence began.

To the extent that the values of $\omega$ given by Eq. (32) are different from those obtained by use of (34), errors will be induced into the system. Values of $\omega$ and $\gamma$ given by Eqs. (34) and (35) are used in Eq. (36), which will subsequently be solved for $e$, $\Phi$, and $\omega_m$.

A solution to Eq. (36) is given previously. A method of obtaining a smooth value of $e$ and $\Phi$ is also given. Further research reveals a more realistic method. Since the previous solution is exact for three measurements (whether the measurements are correct or not), a serious restriction is imposed upon the accuracy of the sensing elements. The improved method indeed requires a minimum of three measure-

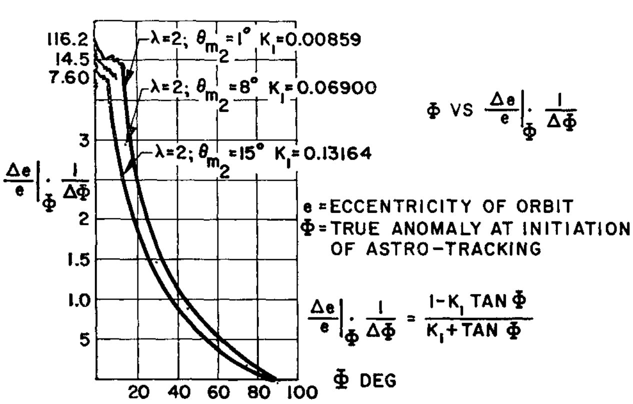


**Fig. 5 Eccentricity error due to computed gyro phase lag error.**

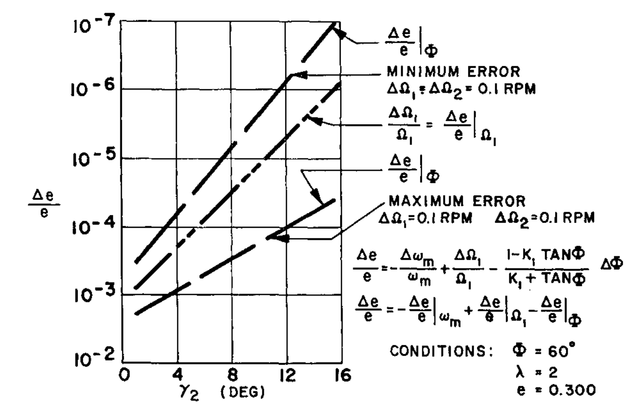


**Fig. 7 Error in eccentricity due to mean gyro speed and phase lag.**

ments or observations, but when more than three observations are used, the equation yields the best value of parameters obtained from observations in a least-squares sense. The additional computer requirements, if any, should be more than compensated by the relaxation of sensor and system performance requirements.

Values of $\omega$ and $\gamma$ from Eqs. (34) and (35) may be stored in a memory unit. For convenience, values will be taken at preset intervals of $\gamma$

$$\gamma_n = n\lambda\Delta\phi \tag{37}$$

where $\lambda$ is any desirable factor ($\lambda = 1$ for all computations in this investigation), $n$ is the number of measurements ($n = 0, 1, 2, 3 \ldots\ldots N$), and $\Delta\phi$ is the interval between observations or data acquisition points. Equation (36) may be written as

$$\omega = \omega_m + A\cos\gamma - B\sin\gamma \tag{38}$$

where

$$A = e_c\omega_m\cos\Phi, \text{ and } B = e_c\omega_m\sin\Phi$$

With $\lambda = 1$ in Eq. (37), and assuming $n$ measurements, Eq. (38) becomes

$$\omega_n = \omega_m + A\cos n\,\Delta\theta - B\sin n\,\Delta\theta \tag{39}$$

Using Eq. (40), we define a function $V_n$ as

$$V_n = \omega_m + A\cos n\,\Delta\theta - B\sin n\,\Delta\theta - \omega_n \tag{40}$$

It has been shown that the normal equations (in a least-squares sense) may be given by[6]

$$\sum_{n=0}^{N} Vn\frac{\partial V_n}{\partial b_k} = 0 \qquad (k = 1, 2, 3) \tag{41}$$

where

$$b_1 = \omega_m \qquad b_2 = A \qquad b_3 = B$$

Using (40) in (41) yields

$$\begin{vmatrix} a_{11} & a_{12} - \tau_1 & -a_{13} - \tau_2 \\ a_{21} & a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{vmatrix} \times \begin{vmatrix} \omega_m - \omega_0 \\ A \\ B \end{vmatrix} = \begin{vmatrix} Z_1 \\ Z_2 \\ Z_3 \end{vmatrix} \tag{42}$$

where

$$\tau_1 = 1/\omega_m[\omega_0 a_{21} - Aa_{22} + 2Ba_{32} + Z_2]$$

$$\tau_2 = 1/\omega_m[\omega_0 a_{31} + Ba_{33} + Z_3]$$

$$a_{11} = N + 1$$

$$a_{22} = \sum_{n=0}^{N}\cos^2 n\Delta\theta = \frac{N}{2} - \frac{1}{4}\left[1 - \frac{\sin 2(N + 1/2)\Delta\theta}{\sin\Delta\theta}\right] + 1$$

$$a_{33} = \sum_{n=0}^{N}\sin^2 n\Delta\theta = \frac{N}{2} + \frac{1}{4}\left[1 - \frac{\sin 2(N + 1/2)\Delta\theta}{\sin\Delta\theta}\right]$$

$$a_{21} = \sum_{n=0}^{N}\cos n\Delta\theta = \frac{1}{2} + \frac{1}{2}\left[\frac{\sin(N + 1/2)\Delta\theta}{\sin(\Delta\theta/2)}\right]$$

$$a_{12} = 2a_{21}$$

$$a_{13} = a_{31} = \sum_{n=0}^{N}\sin n\Delta\theta = \frac{1}{2}\left[(1 - \cos N\Delta\theta)\cot\frac{\Delta\theta}{2} + \sin N\Delta\theta\right]$$

$$a_{32} = a_{23} = \sum_{n=0}^{N}\sin n\Delta\theta\cos n\Delta\theta = \tfrac{1}{4}[(1 - \cos 2N\Delta\theta)\cot\Delta\theta + \sin 2N\Delta\theta]$$

$$Z_1 = -\sum_{j-1}^{N}(N + 1 - j)\Delta\omega_j$$

$$Z_2 = -\sum_{n=1}^{N}\sum_{j=1}^{n}\Delta\omega_j\cos(n\Delta\theta)$$

$$Z_3 = -\sum_{n=1}^{N}\sum_{j=1}^{n}\Delta\omega_j\sin(n\Delta\theta)$$

$$\Delta\omega_j = \omega_j - \omega_{j+1}$$

Solving Eq. (42) by iteration yields[5]

$$\omega_m - \omega_0,\ A,\ B$$

From Eq. (38)

$$e_c = (A^2 + B^2)^{1/2}/\omega_m \tag{43}$$

and

$$\Phi = \cos^{-1}(A/e_c\omega_m) \tag{44}$$

and $P_c$ and $a_c$ are found by Eqs. (22) and (23). The geocentric range becomes

$$r_c = a_c(\omega_m/\omega)(1 - e_c^2) \tag{45}$$

The speed may be written as

$$V_c = \frac{C_1}{a_c}\left[\frac{2\omega/\omega_m}{(1 - e_c^2)} - 1\right]^{1/2} \qquad e < 1 \tag{46}$$

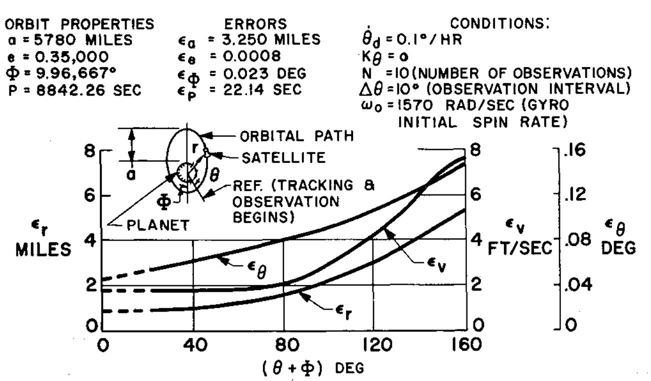


Fig. 8 Relation between $\epsilon_r$, $\epsilon_r$, $\epsilon_\theta$, $(\gamma + \Phi)$ for $\theta_d = 0.1°$hr.

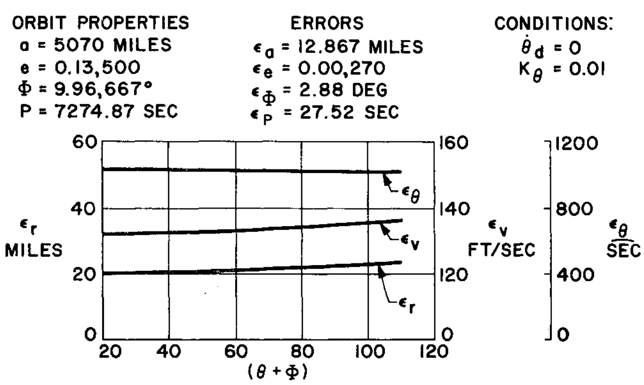


Fig. 9 Relation between $\epsilon_r$, $\epsilon_v$, $\epsilon_\theta$, $(\gamma + \Phi)$ for $K_\theta = 0.01$.

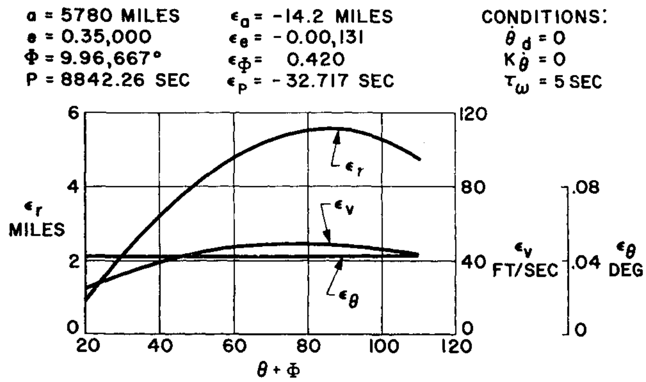


Fig. 10 Relation between $\epsilon_r$, $\epsilon_v$, $\epsilon_\theta$, $(\gamma + \Phi)$ for $\tau_\omega = 5$ sec.

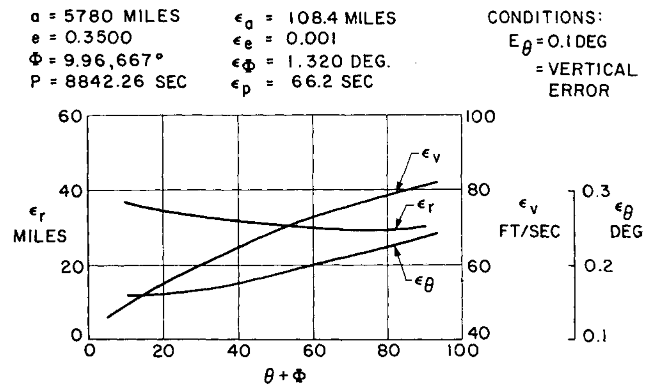


Fig. 11 Relation between $\epsilon_r$, $\epsilon_v$, $\epsilon_\theta$ $(\theta + \Phi)$ for $E_\theta = 0.1°$.

## Vertical Reference

If the orbit-determination system is to have the combined characteristics of being self-contained and nonradiating, a similar requirement must be imposed upon the reference system. For orbits where geocentric radius is not much greater than the radius of the reference planet, the local vertical may be used as an astrogyro reference; otherwise a stellar reference in the orbit plane, or use of the space sextant, may be employed. Presently, the astrogyro is assumed to be on a stable platform whose local vertical is obtained by a horizon scanner. In this orientation, the astrogyro spin axis is required to remain perpendicular to the platform. An electro-optical collimator sensor is used to determine the degree of perpendicularity. The astrogyro is referenced to the stable platform instead of the planet's center; hence, the ability to track the planets center is a function of the platform and error sensor dynamics.

## Solution of Equations

The previously derived equations were solved by use of the 7090 computer. Using this simulation, the effect of various error sources given by Eqs. (27) and (34) were investigated. The theoretical trajectory, or inertial parameters thereof, is obtained by solving Eqs. (6, 7, 23, and 26) with proper initial conditions and planetary areal constant $C_1$. The theoretical parameters are compared with those computed by using the assumed system with the attendent error sources. This comparison yields an error the magnitude of which is proportional to the system component error.

A satellite was assumed to be in a given orbit, the equivalent inertial parameters of which are shown in Fig. 8. The simulated system is assumed to be in the orbiting vehicle, and its task is to compute the orbit. A total of ten observations are made at 10° intervals along the trajectory (assuming the origin to be the earth's center).

During the observation interval, the astrogyro is assumed to be drifting at a rate ($\dot{\theta}_d$) of 0.1 deg/hr. The system is required to determine the orbit under these conditions. Errors in computed parameters are tabulated on Fig. 8. Also shown are the errors in system computed range, velocity, and true anomaly as a function of the actual true anomaly.

Figure 9 gives the properties of a different orbit, and an error of 1% in the measurement of the initial value of $\dot{\theta}$ is assumed. It is quite apparent that the system is very sensitive to this error source. Figure 10 gives similar properties when a 5-sec time lag exists in the gyro spin rate variation control loop.

Another source of error, due to nonverticality, has been evaluated. This was accomplished by adding an error term to Eq. (37) and solving the system of equations again. It was assumed that the platform error is 0.1°. The resulting parameters and orbit properties are shown in Fig. 11.

## Conclusion

The concept and computations presented in this paper demonstrate that a self-contained, nonradiating orbit determination system is feasible. Inertial parameters alone are obtained. The geocentric angle reference is the apse line, or semimajor axis of the ellipse. The orbit will be completely determined when the position of the major axis relative to an earth coordinate axis system is known. At this point, local earth time and stellar reference must be employed.

It is apparent also that the number of equations involving the time parameter are held to a minimum. This is advantageous, since it is known that time warps the space to the extent that the equations are extremely complex. This complexity of course amplifies the computer design requirements.

It is shown that range may be computed to an accuracy that is compatible with values obtainable with more conventional range measurement techniques. The accuracy of satellite speed, geocentric angular position, and orbital period is fair.

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